Involutorial Transformations.

By Frank M. Morgan.

It has been proved by Professor Bertini * that there exist four distinct types of involutorial transformations in a ternary field.

By distinct types he means that no type can be reduced to any other type by means of a series of quadratic transformations.

In the present article I have set up the equations of these four types. The first type, harmonic homology, can be reduced to the form, $\rho x_1 = x'_1$, $\rho x_2 = x'_2$, $\rho x_3 = -x'_3$. It will not be considered further. I have then shown how the transformations of class one and two considered by Bertini,† those of class three and four considered by Martinetti,‡ and those of class five considered by Berzolari,§ may be reduced to some one of the four types. I have then shown by the same process how a transformation of any class may be built.

The four types are:

1) Harmonic homology.

2) Involutorial perspective Jonquières transformations of degree n, with a fixed curve of degree n, which contains an $(n-2)$-fold point.

3) The Geiser transformation of degree eight with seven fundamental points of the third order, and a fixed curve of order six which has double points at the seven fundamental points.

4) The Bertini transformation of degree seventeen with eight fundamental points of the sixth order, and a fixed curve of degree nine, having triple points at the eight fundamental points.

Type II.

To set up a Jonquières transformation, consider a pencil of conics having four distinct basis points, and a pencil of cubics four of whose basis points coincide with the basis points of the conics. This configuration sets up an

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* *Annali di Mat.*, Ser. 2, T. 8, pp. 244–286.

† "Sopra alcune involutioni piane," *Lomb. Ist. Rend.*, Ser. 2, Vol. XVI (1883), pp. 89–101, 190–208.

‡ "Le involuzioni di 3a e 4a classe," *Annali di Mat.*, Ser. 2, T. 12 (1883–1884), pp. 73–106.

§ "Ricerche sulle trasformazioni piane, univoche, involutorie e loro applicazione delle involuzioni di quinta classe," *Annali di Mat.*, Ser. 2, T. 16 (1888–1889), pp. 191–275.

involutorial transformation. For, choose a point P, and pass a conic and a cubic of the pencils through the point P. The conic and the cubic are fixed. Therefore the sixth point of intersection P' is fixed. Conversely, if we choose the point P', the same conic and cubic are fixed, and therefore the point P is determined. That is, the pencil of conics and the pencil of cubics determine an involutorial transformation.

Call the common basis points 1, 2, 3, 4, and the basis points of the cubics, which are not basis points of the conics, 5, 6, 7, 8, 9. The points are the fundamental points of the transformation. The fundamental curves corresponding to 5, 6, 7, 8, 9 are conics. For, take the point 6, with 1, 2, 3, 4. They determine a conic. Take P on this conic, and as the cubics vary for different points P, the remaining sixth point of intersection, which is the image of P, must necessarily be at 6. In the same way the fundamental curves corresponding to 5, 7, 8, 9 are conics.

To find the fundamental curves corresponding to the fundamental points 1, 2, 3, 4, take the pencil of conics $x_1 + u_2 + \lambda (x_2 + u_2') = 0$ and the pencil of cubics $x_1 + v_3 + \lambda (x_2 + v_3') = 0$. That is, we associate curves of the pencil which touch each other at $1 = (0, 0, 1)$.

Eliminating λ, we have

$$(x_1 + u_2)(x_2 + v_3') - (x_1 + v_3)(x_2 + u_2') = 0,$$

i. e.,

$$x_2 u_2 + x_1 v_3' + u_2 v_3' - x_2 v_3 - x_1 u_2' - u_2' v_3 = 0,$$

which is a C_5 having a triple point at 1, double points at 2, 3, 4 and simple points at 5, 6, 7, 8, 9. We will denote this by the following symbol:

$$C_5 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

The fundamental curves corresponding to 2, 3, 4 are likewise quintics. The order of the transformation from the relation, order of Jacobian $= 3(n-1)$, gives $5 + 5 + 5 + 5 + 2 + 2 + 2 + 2 + 2 = 3(n-1)$.

The image of a straight line is a C_{11}, which may be written

$$C_{11} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 5 & 5 & 5 & 5 & 2 & 2 & 2 & 2 & 2 \end{pmatrix}.$$

A C_{11} is fixed by 77 points. The four five-fold points count for 60 and the five double points for 15. Therefore there are two degrees of freedom. The maximum number of double points in a C_{11} is 45. The four five-fold points count as 40 and the five original double points make up the 45.

The whole problem may be approached in a different way. Take the points 2, 3, 4 as the vertices of a fundamental triangle with regard to a quadratic inversion Q. Transforming the pencils of conics and cubics through Q, we obtain a pencil of lines through $1'$, the image of point one, and a pencil of cubics through $1', 2, 3, 4, 5', 6', 7', 8', 9'$. An involutorial transformation is again set up. For, take a point P, and this fixes a line and a cubic of each pencil which intersect in another point P'. Conversely, given P', the same C_1 and C_3 are fixed. Therefore the image of P' is P and the transformation is involutorial.

If the point P is on the line (15), the image of P is the point 5. Every point on (15) goes into 5. Therefore 5 is a fundamental point and (15) a fundamental line. There are therefore eight fundamental lines. The fundamental curve corresponding to the point 1 may be found in the following manner: Consider the C_3 whose equation is

$$X_1 X_3^2 + \Phi_2 (X_1 X_2) X_3 + \Phi_3 (X_1 X_2) = 0,$$

and the C_1

$$X_1 = 0.$$

Also the C_3 whose equation is

$$X_2 X_3^2 + \Phi_2 (X_1 X_2) X_3 + \Phi_3 (X_1 X_2) = 0,$$

and the C_1

$$X_2 = 0.$$

Then form the pencils

$$(X_1 + \lambda X_2) X_3^2 + (\Phi_2 + \lambda \psi_2) X_3 + (\Phi_3 + \lambda \psi_3) = 0$$

and

$$X_1 + \lambda X_2 = 0.$$

Eliminating λ, we get the locus of P,

$$(X_2 \Phi_2 - X_1 \psi_2) X_3 + (\Phi_3 X_2 - \psi_2 X_1) = 0.$$

Therefore the locus is a C_4 with a triple point at 1. The sum of the orders of the fundamental curves equals the order of the Jacobian, which is $3(n-1)$. Therefore

$$8 + 4 = 3(n-1),$$

or $n = 5$.

The C_5 has a four-fold point at 1, for the point P corresponds to P'. Therefore the line $(1P)$ can cut the C_5 only in one point outside of 1, *i. e.*, 1 is a fundamental point. The image of any curve, for example a straight line, will pass through 5. For the line must cut (15) in some point P, and the image of P we saw was 5. For the same reason the curve will pass

through 6, 7, 8, 9. There are certain points which are the images of themselves. These points form the curve of coincident points. It will be called the **Γ** curve.

To determine this curve we proceed as follows: Call the point $1 = (0,0,1)$ and take a point (ξ_1, ξ_2, ξ_3) on the curve. A line through $(0, 0, 1)$ and (ξ_1, ξ_2, ξ_3) is $\xi_1 X_2 - \xi_2 X_1 = 0$. A tangent to the curve at (ξ_1, ξ_2, ξ_3) is

$$X_1 \frac{\delta C_1}{\delta \xi_1} + X_2 \frac{\delta C_2}{\delta \xi_2} + X_3 \frac{\delta C_3}{\delta \xi_3} = 0.$$

These are necessarily identical, for geometrically we see that the curve of coincident points is the locus of the points of tangency of the tangents drawn from $(0, 0, 1)$ to the corresponding curve of the other pencil.

Therefore,

$$\frac{-\xi_2}{\frac{\delta C_3}{\delta \xi_1} + \lambda \frac{\delta C_3'}{\delta \xi_1}} = \frac{\xi_1}{\frac{\delta C_3}{\delta \xi_2} + \frac{\lambda \delta C_3'}{\delta \xi_2}}$$

and

$$\frac{\delta C_3}{\delta \xi_3} + \lambda \frac{\delta C_3'}{\delta \xi_3} = 0,$$

for no term X_3 can appear. Eliminating λ, we have

$$\xi_1 \left| \frac{\delta (C_3 C_3')}{\delta (\xi_1 \xi_2)} \right| = - \xi_2 \left| \frac{\delta (C_3 C_3')}{\delta (\xi_1 \xi_2)} \right|,$$

which is a curve of order five.

Going back to our original cubics

$$C_3 = X_1 X_3^2 + \Phi_2 (X_1 X_2) X_3 + \Phi_3 (X_1 X_2) = 0$$

and

$$C_3 = X_2 X_3^2 + \Phi_2 (X_1 X_2) X_3 + \Phi_3 (X_1 X_2) = 0,$$

getting the derivatives and substituting them in the above equation, gives

$$X_2 \left\{ (2 X_2 X_3 + \psi_2) (\Phi_2 X_3 + \Phi_3) - (2 X_1 X_3 + \Phi_2) \left(\frac{\delta \psi_2}{\delta X_2} + \frac{\delta \psi_3}{\delta X_3} \right) \right\}$$
$$= X_1 \left\{ \left(X_3^2 + \frac{\delta \Phi_2}{\delta X_1} X_2 + \frac{\delta \Phi_3}{\delta X_1} \right) (2 X_2 X_3 + \psi_2) - \left(\frac{\delta \Phi_2}{\delta X_1} X_3 + \frac{\delta \Phi_3}{\delta X_1} \right) \right\}.$$

This curve we see has a triple point at 1.

If we perform the operation $Q\, T_5 Q^{-1}$, we will have the transformation generated by the pencil of conics and cubics.

A C_1 by Q goes into a $C_2\,(2, 3, 4)$. This by T_5 goes into a C_7, the three

fundamental curves corresponding to the points 2, 3, 4 dividing out as factors. This C_7 by Q now goes into a C_{11} with the symbol

$$C_{11}\begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9\\ 5 & 5 & 5 & 5 & 2 & 2 & 2 & 2 & 2\end{pmatrix}.$$

The above results agree exactly with what we had before.

In general, if we have a pencil of lines through the point 1, and a pencil of curves of degree n having an $(n-2)$-fold point at 1, and passing through $2(n-1)$ simple points, it will define an involutorial Jonquières transformation. The curve of coincident points will also be of order n. Any two pencils of curves, one of which, by successive quadratic transformations, may be reduced to a pencil of lines, and the other to a pencil of curves having an $(n-2)$-fold point at the vertex of this pencil of lines, may be solved by the above method.

A very simple manner of obtaining the transformation is to consider it determined by Γ and the pencil of lines. If we choose a point X not on Γ and draw the line $(1X)$, it will cut the curve in two points X_1, X_2. To get the image of X, get the harmonic conjugate to X, X_1, X_2, and call it $\bar{X}$.

Since these four points are harmonic, we have

$$\frac{X-X_1}{X_2-X}=\frac{\bar{X}-X_1}{\bar{X}-X_2},$$

or

$$2X\bar{X}-(X+\bar{X})(X_1+X_2)+2X_1X_2=0. \qquad \text{(A)}$$

The equation of the general C_n having an $(n-2)$-fold point is

$$X^2\Phi_{n-2}(YZ)+X\Phi_{n-1}(YZ)+\Phi_n(YZ)=0,$$

or

$$X^2+X\frac{\Phi_{n-1}}{\Phi_{n-2}}+\frac{\Phi_n}{\Phi_{n-2}}=0.$$

Substituting now in relation (A) gives

$$2X\bar{X}+(X+\bar{X})\frac{\Phi_{n-1}}{\Phi_{n-2}}+2\frac{\Phi_n}{\Phi_{n-2}}=0,$$

or

$$2X\bar{X}\Phi_{n-2}(YZ)+(X+\bar{X})\Phi_{n-1}(YZ)+2\Phi_n(YZ)=0;$$

i. e.,

$$\bar{X}=\frac{X\Phi_{n-1}+2\Phi_n}{-(2X\Phi_{n-2}+\Phi_{n-1})}.$$

But $\bar{Y}=Y$ and $\bar{Z}=Z$; therefore we have

$$\begin{aligned}\bar{X}&=X\Phi_{n-1}+2\Phi_n,\\ \bar{Y}&=-(2X\Phi_{n-2}+\Phi_{n-1})Y,\\ \bar{Z}&=-(2X\Phi_{n-2}+\Phi_{n-1})Z,\end{aligned}$$

which are the equations of the general perspective Jonquières transformation.

A very interesting fact is that if the ratio of the four points is not taken to be harmonic, but is taken to be equal to λ, then the transformation is neither periodic nor birational. Then we have

$$\frac{(X_1-\bar{X})\,(X_2-X)}{(X_1-X)\,(X_2-\bar{X})}=\lambda,$$

or

$$(1-\lambda)\,(X_1X_1+\bar{X}X)-(X_1+X_2)\,(\bar{X}+X)+(1+\lambda)\,(X_1\bar{X}+X_2X)=0.$$

The curve Γ, we saw, was

$$X^2\Phi_{n-2}(YZ)+X\Phi_{n-1}+\Phi_n=0.$$

Therefore

$$X_1X_2=\frac{\Phi_n}{\Phi_{n-2}},$$

$$X_1+X_2=\frac{\Phi_{n-1}}{\Phi_{n-2}}.$$

It then follows that

$$(1-\lambda)\left(\frac{\Phi_n}{\Phi_{n-2}}+\bar{X}X\right)+\frac{\Phi_{n-1}}{\Phi_{n-2}}(\bar{X}+X)+(1+\lambda)\,(X_1\bar{X}+X_2X)=0.$$

If we solve this for X, we see it brings in radicals. Therefore the transformation is neither birational nor periodic.

I will now set up the equations of a Jonquières T_3 which is non-perspective. A non-perspective Jonquières transformation is one such that instead of having the image of a point A on the line connecting it to 1, it is a point B on a separate line $(1B)$. Conversely, the image of B is the point A. The rays through 1 are in involution and therefore contain two fixed rays. These constitute the curve Γ. If two of the fundamental points fall on one of these lines, the order of the transformation will be lowered one unit, and Γ will consist of a line of invariant points and two isolated invariant points. A still more particular case arises when we let two more fundamental points lie on the last fixed line. The curve Γ then consists of four isolated invariant points, and the order of the transformation is again lowered one unit.

In this T_3 whose equations I will derive, I will have one line of invariant points and two isolated invariant points. The invariant line becomes a tangent to the fundamental curve belonging to the point 1, for a tangent to an invariant curve at any fundamental point is tangent to the fundamental curve at that point. For the triangle of reference I will take the triangle whose vertices are 1, 2, 3, calling $(1,2)=X$, $(1,3)=Y$, $(2,3)=Z$. The image of a point

on X is on Y, and conversely, the image of a point on Y is on X. The image of the point 1 is evidently the C_2

$$(X-Y)Z+\lambda XY=0;$$

therefore

$$X'=Y[(X-Y)Z+\lambda XY]$$

and

$$Y=X[(X-Y)Z+\lambda XY].$$

It now remains to find the image of Z. We see 2 goes into Y and 3 into X. We now have to determine what is the image of $(4, 5)$. Let $Y=pX$ be the line connecting 1 to 4, and $Y=qX$ be the line connecting 1 to 5. The line $Y=pX$ is to go into the line $Y=qX$; *i. e.*, $pq=1$.

To find the lines through 1 consider the conic $Z(X-Y)+\lambda XY=0$ and an arbitrary line $AX+BY+CZ=0$. Solving gives

$$-(AX+BY)(X-Y)+\lambda CXY=0.$$

This represents the two lines through 1. We want them to be $Y=pX$ and $Y=qX$. Therefore

$$-AX^2-BXY+AXY+BY^2+\lambda CXY=Y^2-qXY-pXY+pqX^2;$$

i. e.,

$$A=-1 \quad \text{and} \quad B=1.$$

The value of C is indeterminate; so I will call it K. Then

$$Z=(X-Y+KZ)XY.$$

If we form the isologue, we know that $Y-X$ must be a factor. The isologue* is:

$$\begin{vmatrix} Y((X-Y)Z+\lambda YX) & X((X-Y)Z+\lambda XY) & XY(X-Y+KZ) \\ X & Y & Z \\ \alpha & \beta & \gamma \end{vmatrix}$$

If this is to have the factor $Y-X$, it is necessary that $K=\lambda$. Therefore the transformation may be written

$$\frac{X}{Y'Z'(X'-Y'+\lambda Y')}=\frac{Y}{X'Z'(X'-Y'+\lambda Y')}=\frac{Z}{X'Y'(X'-Y'+\lambda Z')}.$$

If one cares to find the two invariant points, he merely has to let $X=X'$, $Y=Y'$, $Z=Z'$ in the above equations, and then solve them, rejecting the factor $Y-X$.

* The definitions of this and of other technical words are given in the long paper of Bertini mentioned above.

This non-perspective Jonquières transformation is immediately transformed into a perspective Jonquières transformation by the linear transformation

$$X = Y', \qquad Y = X', \qquad Z = Z'.$$

That is, we have

$$\frac{X}{X'Z'(X' - Y' + \lambda Y')} = \frac{Y}{Y'Z'(X' - Y' + \lambda Y')} = \frac{Z}{X'Y'(X' - Y' + \lambda Z')}.$$

The curve of coincident points is

$$X'Y'(X' - Y' + \lambda Z') = Z'((X' - Y')Z' + \lambda Y'Z'),$$

which agrees with what we had before.

So in general, any non-perspective Jonquières transformation can be transformed into a perspective Jonquières transformation by a linear transformation which interchanges the rays X and Y and leaves Z unchanged.*

TYPE III.

Let us start with seven basis points, no three of which are on a straight line. Through these seven points we can pass a net of cubic curves. If we choose another point P, this fixes a pencil. The curves of this pencil will intersect in another point Q. Conversely, if we had started with Q, the remaining intersection would have been P. Therefore an involutorial transformation is set up.

Consider the net $a\phi_1 + b\phi_2 + c\phi_3 = 0$ and the two pencils of the net

$$\phi_1 + k\phi_3 = 0, \qquad \phi_2 + l\phi_3 = 0.$$

If the point P is (ξ_1, ξ_2, ξ_3), the two cubics of the pencil become

$$\phi_1\phi_2(\xi) - \phi_2\phi_1(\xi) = 0,$$
$$\phi_2\phi_3(\xi) - \phi_3\phi_2(\xi) = 0.$$

To find the order of the transformation, let us consider what the image of the line $A_1X_1 + A_2X_2 + A_3X_3 = 0$ will be. Solving the last equation for X_1, we see $X_1 = mX_2 + nX_3$. Substituting this in the above equations, we have

$$\phi_1(X_2X_3)\,\phi_2(\xi) - \phi_2(X_2X_3)\,\phi_1(\xi) = 0,$$
$$\phi_2(X_2X_3)\,\phi_3(\xi) - \phi_3(X_2X_3)\,\phi_1(\xi) = 0.$$

Eliminating the X's by Sylvester's method we get a curve of order eight which has a triple point at each of the fundamental points.

The fundamental curves are cubics, for the Jacobian which is of order $3(n-1)$, where n is the order of the transformation, is equal to the sum of the degrees of the fundamental curves. Since all the fundamental curves are

* Dr. Karl Doehlemann: "Geometrische Transformationen," Teil II, p. 169.

associated with symmetrically situated fundamental points, the curves will be the same. Since fundamental curves are uniquely fixed by the fundamental points, the curves will be

$$1:\quad C_3\begin{pmatrix}1&2&3&4&5&6&7\\2&1&1&1&1&1&1\end{pmatrix},$$

$$2:\quad C_3\begin{pmatrix}1&2&3&4&5&6&7\\1&2&1&1&1&1&1\end{pmatrix},$$

$$\cdots\cdots\cdots,$$

$$7:\quad C_3\begin{pmatrix}1&2&3&4&5&6&7\\1&1&1&1&1&1&2\end{pmatrix}.$$

That is, the following transformation is set up:

$$T_8\begin{pmatrix}1&2&3&4&5&6&7\\3&3&3&3&3&3&3\end{pmatrix}.$$

Suppose the two cubics, instead of intersecting in the points P and Q, had the point $P=Q$, *i. e.*, were tangent to each other at P. Such a point as this would be an invariant point, *i. e.*, its image is itself. This means we have

$$a\frac{\delta\phi_1}{\delta\xi_1}+b\frac{\delta\phi_2}{\delta\xi_1}+c\frac{\delta\phi_3}{\delta\xi_1}=0,$$

$$a\frac{\delta\phi_1}{\delta\xi_2}+b\frac{\delta\phi_2}{\delta\xi_2}+c\frac{\delta\phi_3}{\delta\xi_2}=0,$$

$$a\frac{\delta\phi_1}{\delta\xi_3}+b\frac{\delta\phi_2}{\delta\xi_3}+c\frac{\delta\phi_3}{\delta\xi_3}=0.$$

For this to be true the determinant of the coefficients of a, b, c must vanish; *i. e.*,

$$\begin{vmatrix}\frac{\delta\phi_1}{\delta\xi_1}&\frac{\delta\phi_2}{\delta\xi_1}&\frac{\delta\phi_3}{\delta\xi_1}\\[4pt]\frac{\delta\phi_1}{\delta\xi_2}&\frac{\delta\phi_2}{\delta\xi_2}&\frac{\delta\phi_3}{\delta\xi_2}\\[4pt]\frac{\delta\phi_1}{\delta\xi_3}&\frac{\delta\phi_2}{\delta\xi_3}&\frac{\delta\phi_3}{\delta\xi_3}\end{vmatrix}=0.$$

Therefore the curve of coincident points is merely the Jacobian. Since every term is of order two, the determinant is of order six. The multiplicity at a fundamental point is known to be $3i-1$.* In our case $i=1$, or the multiplicity is 2. Therefore the curve of fixed points is

$$T_6\begin{pmatrix}1&2&3&4&5&6&7\\2&2&2&2&2&2&2\end{pmatrix}.$$

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* Doehlemann: "Geometrische Transformationen," Teil II, p. 142.

According to our hypothesis no three of the fundamental points were on a straight line. I will now consider the case when three fundamental points are on a straight line. The curve of order eight, which is the image of a straight line, will necessarily degenerate, for we will have a straight line cutting it in three triple points. The transformation can then be written:

$$T_7\begin{pmatrix}1&2&3&4&5&6&7\\3&3&3&3&2&2&2\end{pmatrix}.$$

The fundamental curves of the points 5, 6, 7 will also contain the line as a factor.

Consider the involutorial inversion whose triangle of reference is 1, 2, 3; then

$$C_1\,I_{123} = C_2\ (1\ 2\ 3),$$
$$C_2\,T_7 = C_5\begin{pmatrix}1&2&3&4&5&6&7\\2&2&2&3&1&1&1\end{pmatrix},$$
$$C_5\,I_{123} = C_4\begin{pmatrix}1&2&3&4'&5'&6'&7'\\1&1&1&3&1&1&1\end{pmatrix}.$$

If the cubics are $\Phi_1 = 0$ and $C_1\,(a\,C_2 + b\,C_2') = 0$, the Jacobian has the factor C_1, or

$$J = C_5\begin{pmatrix}1&2&3&4&5&6&7\\2&2&2&2&1&1&1\end{pmatrix}.$$

Inverting this through I_{123} gives

$$C_4\begin{pmatrix}1&2&3&4'&5'&6'&7'\\1&1&1&2&1&1&1\end{pmatrix}.$$

Therefore the first case reduces to a perspective Jonquières transformation.

Second, suppose that the points 3, 4, 7 were also on a straight line. Then we have a $T_6\begin{pmatrix}1&2&3&4&5&6&7\\3&3&2&2&2&2&1\end{pmatrix}$. If the curves are $\Phi_3 = 0$ and $X_1 X_2\,(a\,X_1 + a\,X_2 - k\,X_3) = 0$, the Jacobian reduces to $C_4\begin{pmatrix}1&2&3&4&5&6&7\\2&2&1&1&1&1&0\end{pmatrix}$. Then

$$C_1\,I_{123} = C_2\ (1\ 2\ 3),$$
$$C_2\,T_6 = C_4\begin{pmatrix}1&2&3&4&5&6&7\\2&2&1&2&1&1&0\end{pmatrix},$$
$$C_4\,I_{123} = C_3\begin{pmatrix}1&2&3&4'&5'&6'&7'\\1&1&0&2&1&1&0\end{pmatrix}.$$

Then Γ reduces to a $C_3\begin{pmatrix}1&2&3&4'&5'&6'&7'\\1&1&0&1&1&1&0\end{pmatrix}$. Therefore this case reduces to a perspective Jonquières transformation.

Third, let the points 2, 4, 6 also be on a straight line. Then the trans-

formation is a $T_5\begin{pmatrix}1&2&3&4&5&6&7\\3&2&2&1&2&1&1\end{pmatrix}$, while I is $C_3\begin{pmatrix}1&2&3&4&5&6&7\\2&1&1&0&1&0&0\end{pmatrix}$. Then

$$C_1\,I_{123} = C_2\,(1\ 2\ 3),$$
$$C_2\,T_5 = C_3\begin{pmatrix}1&2&3&4&5&6&7\\2&1&1&1&1&0&0\end{pmatrix},$$
$$C_3\,I_{123} = C_2\begin{pmatrix}1&2&3&4'&5'&6'&7'\\1&0&0&1&1&0&0\end{pmatrix}.$$

$\boldsymbol{\Gamma}$ becomes $C_2\begin{pmatrix}1&2&3&4'&5'&6'&7'\\1&0&0&0&1&0&0\end{pmatrix}$. Therefore this may be generated from a perspective Jonquières transformation, which in this case is a perspective quadratic transformation.

Fourth, in the original T_8 let the points $1, 2, 7$ and $3, 4, 7$ and $5, 6, 7$ lie on a straight line. Then we have a $T_5\begin{pmatrix}1&2&3&4&5&6&7\\2&2&2&2&2&2&0\end{pmatrix}$, while $\boldsymbol{\Gamma}$ is $C_3\begin{pmatrix}1&2&3&4&5&6&7\\1&1&1&1&1&1&0\end{pmatrix}$. Then

$$C_1\,I_{123} = C_2\,(1\ 2\ 3),$$
$$C_2\,T_5 = C_4\begin{pmatrix}1&2&3&4&5&6&7\\2&2&1&2&1&1&0\end{pmatrix},$$
$$C_4\,I_{123} = C_3\begin{pmatrix}1&2&3&4'&5'&6'&7'\\1&1&0&2&1&1&0\end{pmatrix}.$$

The curve $\boldsymbol{\Gamma}$ remains the same; therefore the transformation may be built from a perspective Jonquières transformation.

Fifth, let the points $1, 2, 7$; $3, 4, 7$; $5, 6, 7$; $2, 4, 6$ be on a straight line. Then we have a $T_4\begin{pmatrix}1&2&3&4&5&6&7\\2&1&2&1&2&1&0\end{pmatrix}$, and $\boldsymbol{\Gamma}$ is $C_2\,(1\ 3\ 5)$. Now

$$C_1\,I_{123} = C_2\,(1\ 2\ 3),$$
$$C_2\,T_4 = C_3\begin{pmatrix}1&2&3&4&5&6&7\\2&1&1&1&1&0&0\end{pmatrix},$$
$$C_3\,I_{123} = C_2\begin{pmatrix}1&2&3&4'&5'&6'&7'\\1&0&0&1&1&0&0\end{pmatrix}.$$

The curve $\boldsymbol{\Gamma}$ becomes $C_2\,(1\ 3\ 5')$. Therefore the transformation may be built from a non-perspective quadratic.

Sixth, let the points $3, 4, 7$; $5, 6, 7$; $2, 4, 6$; $2, 3, 5$ be on straight lines. Then we have a $T_4\begin{pmatrix}1&2&3&4&5&6&7\\3&1&1&1&1&1&1\end{pmatrix}$, where Γ is $C_2\begin{pmatrix}1\\2\end{pmatrix}$. This, we see, is a non-perspective Jonquières transformation in which the curve of coincident points consists of two fixed rays through the point 1.

Seventh, let the points $1, 2, 7$; $3, 4, 7$; $5, 6, 7$: $2, 4, 6$; $2, 3, 5$ be on straight lines. Then we have a $T_3\begin{pmatrix}1&2&3&4&5&6&7\\2&0&1&1&1&1&0\end{pmatrix}$, and $\boldsymbol{\Gamma}$ is $C_2\,(1)$. This is the non-perspective Jonquières transformation whose equations I derived.

The eighth and last possibility is to have 1, 2, 7; 3, 4, 7; 5, 6, 7; 2, 4, 6; 2, 3, 5; 1, 3, 7 on straight lines. We then have a quadratic transformation whose triangle of reference is 1, 4, 5 and whose coincident points are 2, 3, 6, 7.*

TYPE IV.

A curve of order six may have ten double points, and is completely determined by twenty-seven arbitrarily chosen points or by nine arbitrarily chosen double points. If we choose nine double points, the sextic will break up into a cubic counted twice. It is therefore apparent that if nine points are double points, there is some relation between them. Therefore, if eight arbitrarily chosen points are double points on sextic curves, how may the ninth point be situated so that these nine points are on a proper sextic. There can be no proper sextic having the ninth point at the intersection of the two cubics through the eight points, for we would then have a cubic and a sextic intersecting in the nine points and any other point, which makes nineteen intersections of a cubic and a sextic.

Let $\phi(X)=0$ and $f(X)=0$ be the equation of two cubics through eight arbitrarily chosen points, and $\theta(X)=0$ be a proper sextic having a double point at each of these points; then all sextics having these eight double points are of the form

$$\phi^2(X)+\lambda\phi(X)f(X)+\mu f^2(X)+\nu\theta(X)=0.$$

If we impose the condition that the above curve have a ninth double point, we may still have one other point arbitrarily chosen, through which it may pass.

If we pass this curve through a point P', it will be of the form

$$[\phi^2(X)\theta'(X)-\phi'^2(X)\theta(X)]+\mu[f^2(X)\theta'(X)-f'^2(X)\theta(X)]$$
$$+\lambda[\phi(X)f(X)\theta'(X)-\phi'(X)f'(X)\theta(X)=0,$$

where $\phi'(X)$, $f'(X)$, $\theta'(X)$ are the results of substituting the coordinates of the point P' in $\phi(X)$, $f(X)$, $\theta(X)$.

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* de Paolis: "Le trasformazioni piane doppia," *Atti d. r. Accad. d. Lincei,* Series 3ª, Vol. I (1877), pp. 511–544; "La trasformazione piana doppia di secondo ordine, e la sua applicazione alla geometria non euclidea," *Atti d. r. Accad. d. Lincei,* Series 3ª, Vol. II (1878), pp. 31–50; "La trasformazione piana doppia di terzo ordine, primo genere e la sua applicazione alle curve del quarto ordine," *Atti d. r. Accad. d. Lincei,* Series 3ª, Vol. II (1878), pp. 851–878.

Sturm: "Lehre von den geometrischen Verwandtschaften," Vol. IV, pp. 120–122.

Snyder: "The Involutorial Birational Transformation of the Plane, of Order 17," AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXIII (1911), No. 4, pp. 327–336.

This is the equation of a net, and the locus of the double points must be the Jacobian

$$\begin{vmatrix} \phi'^2\theta_1 - 2\phi\phi_1\theta' & f'^2\theta_1 - 2ff_1\theta' & \phi' f'\theta_1 - \phi f_1\theta' - \phi_1 f\theta' \\ \phi'^2\theta_2 - 2\phi\phi_2\theta' & f'^2\theta_2 - 2ff_2\theta' & \phi' f'\theta_2 - \phi f_2\theta' - \phi_2 f\theta' \\ \phi'^2\theta_3 - 2\phi\phi_3\theta' & f'^2\theta_3 - 2ff_3\theta' & \phi' f'\theta_3 - \phi f_3\theta' - \phi_3 f\theta' \end{vmatrix} = 0,$$

i. e.,

$$\theta'(\phi f' - \phi' f)^2 \begin{vmatrix} \phi_1 & f_1 & \theta_1 \\ \phi_2 & f_2 & \theta_2 \\ \phi_3 & f_3 & \theta_3 \end{vmatrix} = 0.$$

We may always choose θ so it does not pass through P'. Therefore θ' does not equal zero. Also no proper sextic could have a ninth double point on the cubic $\phi f' - \phi' f = 0$; for if it did, this cubic would have nineteen intersections with the sextic, eighteen at the nine double points and one at the point P'. The proper locus is therefore

$$K = \begin{vmatrix} \phi_1 & f_1 & \theta_1 \\ \phi_2 & f_2 & \theta_2 \\ \phi_3 & f_3 & \theta_3 \end{vmatrix} = 0.$$

The Jacobian K is of order nine and has a triple point at each of the original points. To show that each of these points is a double point on $K = 0$, we must show that at these points $K_i = 0$ $(i = 1, 2, 3)$. Performing the differentiation, we have

$$K_i = \begin{vmatrix} \phi_{1i} & f_1 & \theta_1 \\ \phi_{2i} & f_2 & \theta_2 \\ \phi_{3i} & f_3 & \theta_3 \end{vmatrix} + \begin{vmatrix} \phi_1 & f_{1i} & \theta_1 \\ \phi_2 & f_{2i} & \theta_2 \\ \phi_3 & f_{3i} & \theta_3 \end{vmatrix} + \begin{vmatrix} \phi_1 & f_1 & \theta_{1i} \\ \phi_2 & f_2 & \theta_{2i} \\ \phi_3 & f_3 & \theta_{3i} \end{vmatrix}.$$

But at these points $\theta_i = 0$; therefore the first two determinants disappear, and we have left the last determinant. Multiplying the three rows by X_1, X_2, X_3 respectively, and adding, we have

$$X_3 K_i = \begin{vmatrix} \phi_1 & f_1 & \theta_{1i} \\ \phi_2 & f_2 & \theta_{2i} \\ 3\phi & 3f & 5\theta_i \end{vmatrix}.$$

At each of these points considered, $\phi = 0$, $f = 0$, $\theta_i = 0$. Therefore, K_i vanishes, *i. e.*, $K = 0$ has a double point. Let us now represent the second derivative of K with respect to X_i and X_k by $K_{ik} = 0$ $(i, k = \text{any two of the numbers } 1, 2, 3)$. We have, neglecting the determinants that have columns vanishing at the points in question,

$$K_{ik} = \begin{vmatrix} \phi_{1i} & f_1 & \theta_{1k} \\ \phi_{2i} & f_2 & \theta_{2k} \\ \phi_{3i} & f_3 & \theta_{3k} \end{vmatrix} + \begin{vmatrix} \phi_1 & f_{1i} & \theta_{1k} \\ \phi_2 & f_{2i} & \theta_{2k} \\ \phi_3 & f_{3i} & \theta_{3k} \end{vmatrix} + \begin{vmatrix} \phi_1 & f_1 & \theta_{1ik} \\ \phi_2 & f_2 & \theta_{2ik} \\ \phi_3 & f_3 & \theta_{3ik} \end{vmatrix} + \begin{vmatrix} \phi_{1k} & f_1 & \theta_{1i} \\ \phi_{2k} & f_2 & \theta_{2i} \\ \phi_{3k} & f_3 & \theta_{3i} \end{vmatrix} + \begin{vmatrix} \phi_1 & f_{1k} & \theta_{1i} \\ \phi_2 & f_{2k} & \theta_{2i} \\ \phi_3 & f_{3k} & \theta_{3i} \end{vmatrix}.$$

Multiplying the rows in each determinant by X_1, X_2 and X_3 respectively, and adding, remembering at these points $\phi = 0$, $f = 0$, $\theta_j = 0$ $(j = 1, 2, 3)$, we have

$$X_3 K_{ik} = 2\,\phi_i \begin{vmatrix} f_1 & \theta_{1k} \\ f_2 & \theta_{2k} \end{vmatrix} - 2\,f_i \begin{vmatrix} \phi_1 & \theta_{1k} \\ \phi_2 & \theta_{2k} \end{vmatrix} + 4\,\theta_{ik} \begin{vmatrix} \phi_1 & f_1 \\ \phi_2 & f_2 \end{vmatrix} + 2\,\phi_k \begin{vmatrix} f_1 & \theta_{1i} \\ f_2 & \theta_{2i} \end{vmatrix} - 2\,f_k \begin{vmatrix} \phi_1 & \theta_{1i} \\ \phi_2 & \theta_{2i} \end{vmatrix};$$

$$\therefore X_3 K_{ik} = 2 \begin{vmatrix} \phi_1 & f_1 & \theta_{1k} \\ \phi_2 & f_2 & \theta_{2k} \\ \phi_3 & f_3 & \theta_{ik} \end{vmatrix} + 2 \begin{vmatrix} \phi_1 & f_1 & \theta_{1i} \\ \phi_2 & f_2 & \theta_{2i} \\ \phi_3 & f_k & \theta_{ik} \end{vmatrix}.$$

But at each of the original eight points, both of these determinants are equal to zero, because either there are two rows alike or, multiplying the rows by X_1, X_2 and X_3 respectively, and adding, we have a row $3\,\phi$, $3\,f$, $5\,\theta_j$ $(j = i$ or $k)$, which vanishes at these points. Therefore, since the second derivative of K vanishes, $K = 0$ has a triple point on this curve.

If we consider the net $\phi^2 + \lambda\,\phi f + \mu\,\theta = 0$ passing through eight double points and one simple point P, this point being the residual intersection of all the cubics of the pencil passing through the eight aforenamed points, we can set up an involutorial transformation, as this accounts for $8 \cdot 3 + 1 = 25$ constants and 27 constants determine a sextic curve. Therefore take any point Q and pass a sextic of each pencil through it. These will intersect in another point R. Conversely, starting with R, the remaining intersection will be Q, *i. e.*, we have an involutorial transformation.

The fundamental curves may be found as follows: Suppose Q, the arbitrarily chosen point, to coincide with a fundamental point. Then $\phi = 0$ and $f = 0$ would be fixed cubics, and the corresponding point P would be any point on the fixed curve which is a C_6 having the symbol

$$C_6\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \end{pmatrix}.$$

Similarly, for 2 we have

$$C_6\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 2 & 2 & 2 & 2 & 2 & 2 \end{pmatrix},$$

$$\cdots\cdots\cdots,$$

and for 8 we have

$$C_6\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 3 \end{pmatrix}.$$

The order of the transformation is therefore easily obtained by remembering that $3\,(n-1)$ equals the sum of the orders of the fundamental curves. Therefore $3\,(n-1) = 8 \cdot 6$ or $n = 17$.

The curve of coincident points is evidently the Jacobian which we have found. Therefore we have an involutorial transformation of order seventeen

with eight fundamental points, which have corresponding fundamental curves of order six, that have a triple point at the fundamental point to which they correspond, and have double points at the other fundamental points. There is, however, a ninth fundamental point of a different type. If we try to find the image of the point which is the residual intersection of all the cubics through the points 1, 2, 3, 4, 5, 6, 7, 8, we see it is itself. But, however, we know that the curve of coincident points does not go through this point. We have then an isolated invariant point. The transformation is therefore

$$T_{17}\begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9\\ 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 0\end{pmatrix}.$$

The particular cases of three points on a line, or five points on a conic, may be reduced to a Geiser transformation by the following scheme:

Let the points 1, 2, 3 be on a straight line. Since in T_{17} a general line not passing through a fundamental point goes into a C_{17} with 8 P_6, it follows that if 1, 2, 3 lie on a line, it must divide out as a factor. The symbol of the transformation may now be written

$$C_1\,T_{16} = C_{16}\begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9\\ 5 & 5 & 5 & 6 & 6 & 6 & 6 & 6 & 0\end{pmatrix}.$$

The fundamental curves of 1, 2, 3 must also contain the line 1, 2, 3 as a factor, becoming for 1, $C_5\begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6 & 7 & 8\\ 2 & 1 & 1 & 2 & 2 & 2 & 2 & 2\end{pmatrix}$, and similarly for the others. To see if T_{16} can be reduced to a simpler form, consider a set of points $1', 2', 3', 4', 5'$, 6, 7, 8 such that $1', 2', 3'$, 6, 7, 8 lie on a conic. Consider the involutorial inversion I_{678}. The conic will go into a straight line through 1, 2, 3. Let $4', 5'$ go into 4, 5. Then, by I_{678}, an arbitrary straight line will go into a conic through 6, 7, 8; *i. e.*,

$$C_1\,I_{678} = C_2\ (6\ 7\ 8).$$

Now transform this conic by T_{16},

$$C_2\,T_{16} = C_{14}\begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6 & 7 & 8\\ 4 & 4 & 4 & 6 & 6 & 5 & 5 & 5\end{pmatrix}.$$

Then transform this C_{14} by I_{678} backwards,

$$C_{14}\,I_{678}^{-1} = C_{13}\begin{pmatrix}1' & 2' & 3' & 4' & 5' & 6 & 7 & 8 & 9\\ 4 & 4 & 4 & 6 & 6 & 4 & 4 & 4 & 0\end{pmatrix}.$$

Therefore

$$I\,T_{16}\,I^{-1} = T_{13}.$$

It now is necessary to obtain the configuration of the fundamental curves of T_{13}. No new fundamental points were introduced. We proceed as follows:

The point $1'$ goes into 1 by I. Then 1 by T_{16} goes into

$$C_5\begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6 & 7 & 8\\ 2 & 1 & 1 & 2 & 2 & 2 & 2 & 2\end{pmatrix}.$$

This by I goes into

$$C_4\begin{pmatrix}1' & 2' & 3' & 4' & 5' & 6 & 7 & 8\\ 2 & 1 & 1 & 2 & 2 & 1 & 1 & 1\end{pmatrix}.$$

Thus

$$1:\quad C_4\begin{pmatrix}1' & 2' & 3' & 4' & 5' & 6 & 7 & 8\\ 2 & 1 & 1 & 2 & 2 & 1 & 1 & 1\end{pmatrix}. \qquad 2:\quad C_4\begin{pmatrix}1' & 2' & 3' & 4' & 5' & 6 & 7 & 8\\ 1 & 2 & 1 & 2 & 2 & 1 & 1 & 1\end{pmatrix}.$$

$$3:\quad C_4\begin{pmatrix}1' & 2' & 3' & 4' & 5' & 6 & 7 & 8\\ 1 & 1 & 2 & 2 & 2 & 1 & 1 & 1\end{pmatrix}. \qquad 4:\quad C_6\begin{pmatrix}1' & 2' & 3' & 4' & 5' & 6 & 7 & 8\\ 2 & 2 & 2 & 3 & 2 & 2 & 2 & 2\end{pmatrix}.$$

$$5:\quad C_6\begin{pmatrix}1' & 2' & 3' & 4' & 5' & 6 & 7 & 8\\ 2 & 2 & 2 & 2 & 3 & 2 & 2 & 2\end{pmatrix}. \qquad 6:\quad C_4\begin{pmatrix}1' & 2' & 3' & 4' & 5' & 6 & 7 & 8\\ 1 & 1 & 1 & 2 & 2 & 2 & 1 & 1\end{pmatrix}.$$

$$7:\quad C_4\begin{pmatrix}1' & 2' & 3' & 4' & 5' & 6 & 7 & 8\\ 1 & 1 & 1 & 2 & 2 & 1 & 2 & 1\end{pmatrix}. \qquad 8:\quad C_4\begin{pmatrix}1' & 2' & 3' & 4' & 5' & 6 & 7 & 8\\ 1 & 1 & 1 & 2 & 2 & 1 & 1 & 2\end{pmatrix}.$$

The next step will be to reduce T_{13}. It will be recalled that $1', 2', 3', 6, 7, 8$ lie on a conic. Take $4', 5', 6$ as the vertices of an inverting triangle, and five other points, $1'', 2'', 3'', 7'', 8''$, which will go into $1', 2', 3', 7', 8'$ by $I_{4'5'6}$. An arbitrary line will go into the conic $C_2\,(4', 5', 6)$; *i. e.*,

$$C_1\, I_{456} = C_2\,(4\ 5\ 6).$$

This curve will go into a curve of order ten by T_{13} having the signature

$$C_{10}\begin{pmatrix}1' & 2' & 3' & 4' & 5' & 6 & 7 & 8 & 9\\ 3 & 3 & 3 & 5 & 5 & 2 & 3 & 3 & 0\end{pmatrix}.$$

This curve is now to be transformed by $I^{-1}_{4'5'6}$. It becomes

$$C_8\begin{pmatrix}1'' & 2'' & 3'' & 4' & 5' & 6 & 7'' & 8'' & 9\\ 3 & 3 & 3 & 3 & 3 & 0 & 3 & 3 & 0\end{pmatrix}.$$

To get the configuration of the fundamental points proceed as before. $1''$ by $I_{4'5'6}$ goes into $1'$. Then $1'$ by T_{13} is $C_4\begin{pmatrix}1' & 2' & 3' & 4' & 5' & 6 & 7 & 8\\ 2 & 1 & 1 & 2 & 2 & 1 & 1 & 1\end{pmatrix}$. This by $I_{4'5'6}$ becomes $C_3\begin{pmatrix}1'' & 2'' & 3'' & 4' & 5' & 6 & 7'' & 8''\\ 2 & 1 & 1 & 1 & 1 & 0 & 1 & 1\end{pmatrix}$, and similarly for the others. But this is exactly the configuration of the general Geiser which we have considered. Having considered the special cases of the Geiser, there is nothing more to be considered in this transformation.*

Bertini † has considered the transformations of Classes I and II. By

* Sturm: "Lehre von den geometrischen Verwandtschaften," Vol. IV, pp. 120–122.

Snyder: "The Involutorial Birational Transformation of the Plane, of Order 17," AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXIII (1911), No. 4, pp. 327–336.

† "Sopra alcune involutioni piane," *Lomb. Ist. Rend.*, Ser. 2, Vol. XVI (1883), pp. 89–101.

class is merely meant this: Let n be the order of the transformation and m the degree of Γ; then the class of the transformation is $\frac{n-m}{2}$. The class is evidently not an invariant property, for we may transform so as to keep Γ invariant and not the order of the transformation. The transformations of Class I are the first five special cases of the Geiser which have already been considered. By the same process the transformations of Class II all reduce to one of the four types.

To amplify this method, I will consider a few transformations of Classes III and IV studied by Martinetti * and one of the class studied by Berzolari. †

Consider a pencil of lines 1 and a pencil of conics 2, 3, 4, 5. Let a line of 1 and a conic 2, 3, 4, 5 be determined by a point P. They intersect in one other point Q. Conversely, fix Q and you get the point P. Therefore we have a one-to-one correspondence. The fundamental curves are

1) $C_2(12345)$, 2) $C_1(12)$, 3) $C_1(13)$, 4) $C_1(14)$, 5) $C_1(15)$.

Using the relation $3(n-1) =$ sum of orders of fundamental curves, we have $n = 3$. Therefore we have an involutorial transformation $T_3\begin{pmatrix}1&2&3&4&5\\2&1&1&1&1\end{pmatrix}$. Performing the operation $I_{236}\,T_3\,I_{236}^{-1}$, we have

$$C_1\,I_{236} = C_2\,(2\;3\;6),$$
$$C_2\,T_3 = C_4\begin{pmatrix}1&2&3&4&5&6'\\2&1&1&2&2&1\end{pmatrix},$$
$$C_4\,I_{236}^{-1} = C_6\begin{pmatrix}1'&2&3&4'&5'&6&6''\\2&3&3&2&2&2&1\end{pmatrix}.$$

The pencil of lines becomes a pencil of conics, and the pencil of conics remains a pencil of conics. We now have, then, an involutorial transformation defined by two pencils of conics having two basis points 1 and 2 in common, and the curves of the first pencil also passing through 3, 4, while those of the second pencil pass through 5, 6. Rewriting, we will call the transformation

$$T_6\begin{pmatrix}1&2&3&4&5&6&7\\3&3&2&2&2&2&1\end{pmatrix}.$$

Then

$$C_1\,I_{189} = C_2\;(1\;8\;9),$$
$$C_2\,T_6 = C_9\begin{pmatrix}1&2&3&4&5&6&7&8'&9'\\4&5&3&3&3&3&1&1&1\end{pmatrix},$$
$$C_9\,I_{189}^{-1} = C_{14}\begin{pmatrix}1&2&3&4&5&6&7&8&9&8''&9''\\9&5&3&3&3&3&1&5&5&1&1\end{pmatrix}.$$

* "Le involutioni di 3a e 4a classe," *Annali di Mat.*, Ser. 2, T. 12 (1883, 1884), pp. 73–106.

† "Ricerche sulle trasformazioni piane, univoche, involutorie e loro applicazione alla determinazione delle involuzioni di quinta classe, *Annali di Mat.*, Ser. 2, T. 16 (1888, 1889), pp. 191–275.

The Γ of the T_3 is $C_3\begin{pmatrix}1\,2\,3\,4\,5\\1\,1\,1\,1\,1\end{pmatrix}$. Transforming this by I_{236} gives $C_4\begin{pmatrix}1\,2\,3\,4\,5\,6\\1\,2\,2\,1\,1\,1\end{pmatrix}$. Thus we see that the T_6 is of Class I. Rewriting, as we did with the transformation, we have Γ equal to $C_4\begin{pmatrix}1\,2\,3\,4\,5\,6\\2\,2\,1\,1\,1\,1\end{pmatrix}$. Transforming this by I_{189}, we have $C_6\begin{pmatrix}1\,2\,3\,4\,5\,6\,7\,8\,9\\4\,2\,1\,1\,1\,1\,0\,2\,2\end{pmatrix}$. Therefore the T_{14} is of Class IV.

By letting certain points lie on straight lines, we can have particular cases of the above transformation. For example, if 2, 3, 5 are on a line in the T_6, we have a T_5. If this is transformed by I_{189}, as before, we get a T_{10}, which is of Class III.

Martinetti,* in discussing the transformations of Class III, gives as his last species $T_6\begin{pmatrix}1\,2\,3\,4\,5\,6\,7\,8\,9\,10\,11\\3\,3\,2\,2\,2\,2\,1\,0\,0\,0\,0\end{pmatrix}$, which is generated, he says, by $C_2\,(1\,2\,3\,4)$ and $C_2\,(1\,2\,3\,5)$. Berzolari in the *Annali di Mat.*, Ser. 2, Tomo 16, p. 201, shows that this transformation does not exist. This can easily be shown by the method used in the previous problems. If we invert the two pencils by I_{123}, we get two pencils of lines. These generate a T_2. If this is transformed by I_{123}, we ought to arrive at the above transformation. But $I_{123}\,T_2\,I_{123}^{-1}$ gives T_8.

A very interesting transformation of Class II may be set up in the following manner: Consider the T_3 generated by $C_1\,(5)$ and $C_2\,(1\,2\,3\,4)$.

$$C_1\,I_{167} = C_2\,(1\,6\,7),$$
$$C_2\,T_3 = C_5\begin{pmatrix}1\,2\,3\,4\,5\,6\,7\\1\,2\,2\,2\,3\,1\,1\end{pmatrix},$$
$$C_5\,I_{167}^{-1} = C_7\begin{pmatrix}1\,2\,3\,4\,5\,6\,7\\3\,2\,2\,2\,3\,3\,3\end{pmatrix}.$$

The fundamental curve corresponding to

$$1:\ C_3\begin{pmatrix}1\,2\,3\,4\,5\,6\,7\\1\,1\,1\,1\,2\,1\,1\end{pmatrix},\qquad 5:\ C_3\begin{pmatrix}1\,2\,3\,4\,5\,6\,7\\2\,1\,1\,1\,1\,1\,1\end{pmatrix},$$
$$6:\ C_3\begin{pmatrix}1\,2\,3\,4\,5\,6\,7\\1\,1\,1\,1\,1\,1\,2\end{pmatrix},\qquad 7:\ C_3\begin{pmatrix}1\,2\,3\,4\,5\,6\,7\\1\,1\,1\,1\,1\,2\,1\end{pmatrix}.$$

It will be noticed that the image of the lines (1 7) (5 6) (7 5) (1 6) are the lines themselves. Therefore the points of intersection of (1 7) (5 6) and (1 6) (7 5) are invariant. The transformation therefore is

$$T_7\begin{pmatrix}1\,2\,3\,4\,5\,6\,7\,8\,9\\3\,2\,2\,2\,3\,3\,3\,0\,0\end{pmatrix},$$

while Γ is $C_3\,(1\,2\,3\,4\,5\,6\,7)$.

If T_7 is transformed by I_{2310}, we get a T_{14} of Class V.

* "Le involuzioni di 3ª e 4ª classe, *Annali di Mat.*, Ser. 2, T. 12 (1883–1884), pp. 73–106.

I will now show how to build transformations of Class v from the general Jonquières transformation. The same procedure may be applied to the Geiser and to the Bertini. In fact, it may be applied to any involutorial transformation already found, so as to get one of higher class. The general Jonquières transformation is

$$T_n \begin{pmatrix} 1 & 2\,3\,4\,5 \ldots\ldots & 2n-1 \\ n-1 & 1\,1\,1\,1 \ldots\ldots & 1 \end{pmatrix}.$$

I. Consider the quadratic inversion I_{234}. Then

$$C_1\, I_{234} = C_2\ (2\ 3\ 4),$$

$$C_2\, T_n = C_{2n-3} \begin{pmatrix} 1 & 2\,3\,4\,5 \ldots\ldots & 2n-1 \\ 2n-1 & 1\,1\,1\,1 \ldots\ldots & 2 \end{pmatrix},$$

$$C_{2n-3}\, I_{234}^{-1} = C_{4n-9} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \ldots\ldots & 2n-1 \\ 2n-5 & 2n-5 & 2n-5 & 2n-5 & 2 \ldots\ldots & 2 \end{pmatrix}.$$

Γ becomes

$$C_{2n-3} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \ldots\ldots & 2n-1 \\ n-2 & n-2 & n-2 & n-2 & 1 \ldots\ldots & 1 \end{pmatrix}.$$

Therefore $\nu = n-3$. The fundamental curves are

$$1: \quad C_{2n-5} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \ldots\ldots & 2n-1 \\ n-2 & n-3 & n-3 & n-3 & 1 \ldots\ldots & 1 \end{pmatrix},$$

$$2: \quad C_{2n-5} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \ldots\ldots & 2n-1 \\ n-3 & n-2 & n-3 & n-3 & 1 \ldots\ldots & 1 \end{pmatrix},$$

. . . . ,

$$5: \quad C_2\ (1\ 2\ 3\ 4\ 5),$$

. . . . ,

$$2n-1: \quad C_2\ (1\ 2\ 3\ 4\ 2n-1).$$

Writing this transformation in terms of v, we have

$$T_{4v+3} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5\ 6 \ldots\ldots & 2n-1 \\ 2v+1 & 2v+1 & 2v+1 & 2v+1 & 2\ 2 \ldots\ldots & 2 \end{pmatrix}.$$

Γ is
$$C_{2v+3} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \ldots\ldots & 2n-1 \\ v+1 & v+1 & v+1 & v+1 & 1 \ldots\ldots & 1 \end{pmatrix}.$$

The transformation has the following fundamental curves:

$$\begin{pmatrix} 1 & 2\,3\,4\,5 \ldots\ldots & 2v+5 \\ v+1 & v\,v\,v\,1 \ldots\ldots & 1 \end{pmatrix}^{1}_{2v+1} \quad \begin{pmatrix} 1 & 2 & 3\,4\,5 \ldots\ldots & 2v+5 \\ v & v+1 & v\,v\,1 \ldots\ldots & 1 \end{pmatrix}^{2}_{2v+1},$$

. . . . ,

$$(1\ 2\ 3\ 4\ 5)^5_2\ (1\ 2\ 3\ 4\ 6)^6_2 \ldots\ldots (1\ 2\ 3\ 4\ 2v+5)^{2v+5}_2.$$

The number to the right and above the parenthesis is the fundamental point to which the curve belongs, while the number below the last named number gives the order of the fundamental curve.

II. Transform the general Jonquières transformation by I_{23P}, where the point P is any point except a point on Γ. Then

$$C_1 I_{23P} = C_2\,(2\ 3\ P),$$

$$C_2 T_n = C_{2n-2}\begin{pmatrix} 1 & 2\ 3\ 4\ 5 \ldots\ldots & 2n-1 & P' \\ 2n-4 & 1\ 1\ 2\ 2 \ldots\ldots & 2 & 1 \end{pmatrix},$$

$$C_{2n-2} I_{23P}^{-1} = C_{4n-6}\begin{pmatrix} 1 & 2 & 3 & 4\ 5 \ldots\ldots & 2n-1 & P'' & P \\ 2n-4 & 2n-3 & 2n-3 & 2\ 2 \ldots\ldots & 2 & 1 & 2n-4 \end{pmatrix}.$$

Γ becomes

$$C_{2n-2}\begin{pmatrix} 1 & 2 & 3 & 4\ 5 \ldots\ldots & 2n-1 & P \\ n-2 & n-1 & n-1 & 1\ 1 \ldots\ldots & 1 & n-2 \end{pmatrix}.$$

Therefore the class $v = n-2$.

Writing everything in terms of v and calling the points $1\ 2\ 3\ 4\ 5 \ldots\ldots 2v+5$, we have

$$T_{4v+2}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \ldots\ldots & 2v+4 & 2v+5 \\ 2v+1 & 2v+1 & 2v & 2v & 2 \ldots\ldots & 2 & 1 \end{pmatrix},$$

Γ is

$$C_{2v+2}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \ldots\ldots & 2v+4 \\ v+1 & v+1 & v & v & 1 \ldots\ldots & 1 \end{pmatrix}.$$

The fundamental curves are

$$\begin{pmatrix} 1 & 2\ 3\ 4\ 5 \ldots\ldots & 2v+5 \\ v+1 & v\ v\ v\ 1 \ldots\ldots & 1 \end{pmatrix}^1_{2v+1} \quad \begin{pmatrix} 1 & 2 & 3\ 4\ 5 \ldots\ldots & 2v+5 \\ v & v+1 & v\ v\ 1 \ldots\ldots & 1 \end{pmatrix}^2_{2v+1},$$

$$\begin{pmatrix} 1\ 2\ 3 & 4 & 5 \ldots\ldots & 2v+4 \\ v\ v\ v & v-1 & 1 \ldots\ldots & 1 \end{pmatrix}^3_{2v} \quad \begin{pmatrix} 1\ 2 & 3 & 4\ 5 \ldots\ldots & 2v+4 \\ v\ v & v-1 & v\ 1 \ldots\ldots & 1 \end{pmatrix}^4_{2v},$$

$$(1\ 2\ 3\ 4\ 5)^5_2\ (1\ 2\ 3\ 4\ 6)^6_2 \ldots\ldots (1\ 2\ 3\ 4\ 2v+4)^{2v+4}_2\ (1\ 2)^{2v+5}_1.$$

It will be noticed that this is equivalent to the first transformation when the points $3, 4, 2v+5$ are on a straight line.

III. Transform the Jonquières transformation by I_{2QP}, where Q and P are points that are not on Γ. Then

$$C_1 I_{2QP} = C_2\,(2\ Q\ P),$$

$$C_2 T_n = C_{2n-1}\begin{pmatrix} 1 & 2\ 3\ 4 \ldots\ldots & 2n-1 & Q' & P' \\ 2n-3 & 1\ 2\ 2 \ldots\ldots & 2 & 1 & 1 \end{pmatrix},$$

$$C_{2n-1} I_{2QP} = C_{4n-3}\begin{pmatrix} 1 & 2 & 3\ 4 \ldots\ldots & 2n-1 & Q'' & P'' & Q & P \\ 2n-3 & 2n-1 & 2\ 2 \ldots\ldots & 2 & 1 & 1 & 2n-2 & 2n-2 \end{pmatrix}.$$

Γ is

$$C_{2n-1}\begin{pmatrix} 1 & 2\ 3\ 4 \ldots\ldots & 2n-1 & Q & P \\ n-2 & n\ 1\ 1 \ldots\ldots & 1 & n-1 & n-1 \end{pmatrix}.$$

Therefore $v = n-1$.

In terms of v we have, rearranging the order of the points,

$$T_{4v+1}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & 2v+3 & 2v+4 & 2v+5 \\ 2v+1 & 2v & 2v & 2v-1 & 2 & \dots & 2 & 1 & 1 \end{pmatrix}.$$

Γ is

$$C_{2v+1}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & 2v+3 \\ v+1 & v & v & v-1 & 1 & \dots & 1 \end{pmatrix}.$$

The fundamental curves are

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & 2v+5 \\ v+1 & v & v & v & 1 & \dots & 1 \end{pmatrix}^1_{2v+1} \quad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & 2v+3 & 2v+4 \\ v & v & v & v-1 & 1 & \dots & 1 & 1 \end{pmatrix}^2_{2v}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & 2v+3 & 2v+5 \\ v & v & v & v-1 & 1 & \dots & 1 & 1 \end{pmatrix}^3_{2v} \quad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & 2v+3 \\ v & v-1 & v-1 & v-1 & 1 & \dots & 1 \end{pmatrix}^4_{2v-1}$$

$$(1\ 2\ 3\ 4\ 5)^5_2\ (1\ 2\ 3\ 4\ 6)^6_2 \dots (1\ 2\ 3\ 4\ 2v+3)^{2v+3}_2\ (1\ 2)^{2v+4}_1\ (1\ 3)^{2v+5}_1.$$

Notice that this transformation is equivalent to having $(3\ 4\ 2v+5)$ $(2\ 4\ 2v+4)$ of the original transformation, in a line.

IV. Transform the Jonquières transformation now through $I_{23\bar{P}}$, where $\bar{P}$ is a point on Γ. This transformation is of interest, for we will see that taking P on the invariant curve is equivalent to taking out two linear factors of I. Furthermore, we will see that the transformation will have an invariant point which is not a point on Γ. The location of this point will also be shown. Now

$$C_1 I_{23\bar{P}} = C_2\,(2\ 3\ \bar{P}),$$

$$C_2 T_n = C_{2n-2}\begin{pmatrix} 1 & 2 & 3 & 4 & \dots & 2n-1 & \bar{P} \\ 2n-4 & 1 & 1 & 2 & \dots & 2 & 1 \end{pmatrix},$$

$$C_{2n-2} I_{23\bar{P}} = T_{4n-7}\begin{pmatrix} 1 & 2 & 3 & 4 & \dots & 2n-1 & \bar{P} \\ 2n-4 & 2n-4 & 2n-4 & 2 & \dots & 2 & 2n-4 \end{pmatrix}.$$

Then Γ is

$$C_{2n-3}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & 2n-1 & \bar{P} \\ n-2 & n-2 & n-2 & 1 & 1 & \dots & 1 & n-2 \end{pmatrix}$$

and $v = n-2$.

Writing the transformation in terms of v, we have

$$T_{4v+1}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & \dots & 2v+4 \\ 2v & 2v & 2v & 2v & 2 & 2 & \dots & 2 \end{pmatrix}.$$

Γ is

$$C_{2v+1}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & 2v+4 \\ v & v & v & v & 1 & \dots & 1 \end{pmatrix}.$$

The fundamental curves are

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & 2v+4 \\ v & v-1 & v & v & 1 & \dots & 1 \end{pmatrix}^1_{2v} \quad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & 2v+4 \\ v-1 & v & v & v & 1 & \dots & 1 \end{pmatrix}^2_{2v}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & \dots & 2v+4 \\ v & v & v & v-1 & 1 & 1 & \dots & 1 \end{pmatrix}^3_{2v} \quad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & 2v+4 \\ v & v & v-1 & v & 1 & \dots & 1 \end{pmatrix}^4_{2v}$$

$$(1\ 2\ 3\ 4\ 5)^5_2 \dots (1\ 2\ 3\ 4\ 2v+4)^{2v+4}_2.$$

It will be noticed that the image of (1 2) is itself. The same is true of (3 4). Therefore the point of intersection of these two lines is invariant. The transformation therefore is

$$T_{4v+1}\begin{pmatrix}1 & 2 & 3 & 4 & 5 & \ldots. & 2v+4 & 2v+5\\ 2v & 2v & 2v & 2v & 2 & \ldots. & 2 & 0\end{pmatrix}.$$

This is equivalent to I where $(3\ 4\ 2v+5)$ $(1\ 2\ 2v+5)$ are on a line.

V. Transform by $I_{2\bar{Q}P}$, where Q is a point on Γ and P is not. Then

$$C_1 I_{2\bar{Q}P} = C_2\,(2\ \bar{Q}\ P),$$

$$C_2 T_n = C_{2n-1}\begin{pmatrix}1 & 2\ 3\ 4 & \ldots. & 2n-1 & \bar{Q} & P'\\ 2n-3 & 1\ 2\ 2 & \ldots. & 2 & 1 & 1\end{pmatrix},$$

$$C_{2n-1} I_{2\bar{Q}P} = C_{4n-4}\begin{pmatrix}1 & 2 & 3\ 4 & \ldots. & 2n-1 & \bar{Q} & P'' & P\\ 2n-3 & 2n-2 & 2\ 2 & \ldots. & 2 & 1 & 2n-2 & 2n-3\end{pmatrix}.$$

Γ is
$$C_{2n-2}\begin{pmatrix}1 & 2\ 3\ 4\ 5 & \ldots. & 2n-1 & \bar{Q} & P\\ n-2 & n\ 1\ 1\ 1 & \ldots. & 1 & n-1 & n-1\end{pmatrix}$$
and $v = n-1$.

In terms of v, we have

$$T_{4v}\begin{pmatrix}1 & 2 & 3 & 4 & 5 & \ldots. & 2v+3 & 2v+4\\ 2v & 2v & 2v-1 & 2v-1 & 2 & \ldots. & 2 & 1\end{pmatrix}.$$

Γ is
$$C_{2v}\begin{pmatrix}1 & 2 & 3 & 4 & 5 & \ldots. & 2v+3\\ v & v & v-1 & v-1 & 1 & \ldots. & 1\end{pmatrix}.$$

The fundamental curves are

$$\begin{pmatrix}1 & 2 & 3 & 4\ 5 & \ldots. & 2v+4\\ v & v & v-1 & v\ 1 & \ldots. & 1\end{pmatrix}^1_{2v} \quad \begin{pmatrix}1\ 2\ 3 & 4 & 5 & \ldots. & 2v+4\\ v\ v\ v & v-1 & 1 & \ldots. & 1\end{pmatrix}^2_{2v}$$

$$\begin{pmatrix}1 & 2 & 3 & 4 & 5 & \ldots. & 2v+3\\ v-1 & v & v-1 & v-1 & 1 & \ldots. & 1\end{pmatrix}^3_{2v-1} \begin{pmatrix}1 & 2 & 3 & 4 & 5 & \ldots. & 2v+3\\ v & v-1 & v-1 & v-1 & 1 & \ldots. & 1\end{pmatrix}^4_{2v-1}$$

$$(1\ 2\ 3\ 4\ 5)^5_2 \ \ldots.\ (1\ 2\ 3\ 4\ 2v+3)^{2v+3}_2\ (1\ 2)^{2v+4}_1.$$

It will be noticed that the lines (1 3) and (2 4) are invariant. Therefore the transformation is

$$T_{4v}\begin{pmatrix}1 & 2 & 3 & 4 & 5 & \ldots. & 2v+3 & 2v+4 & 2v+5\\ 2v & 2v & 2v-1 & 2v-1 & 2 & \ldots. & 2 & 1 & 0\end{pmatrix}.$$

It will be noticed that this is equivalent to I with three factors taken out. One goes out on account of the choice of P and two for $\bar{Q}$, since $\bar{Q}$ is on Γ.

VI. Invert by I_{PQR}, where PQR are points not on Γ. Then

$$C_1 I_{PQR} = C_2\,(P\ Q\ R),$$

$$C_2 T_n = C_{2n}\begin{pmatrix}1 & 2\ 3 & \ldots. & 2n-1 & P' & Q' & R'\\ 2n-2 & 2\ 2 & \ldots. & 2 & 1 & 1 & 1\end{pmatrix},$$

$$C_{2n} I_{PQR} = C_{4n}\begin{pmatrix}1 & 2\ 3 & \ldots. & 2n-1 & P'' & Q'' & R'' & P & Q & R\\ 2n-2 & 2\ 2 & \ldots. & 2 & 1 & 1 & 1 & 2n & 2n & 2n\end{pmatrix}.$$

Γ is
$$C_{2n}\begin{pmatrix} 1 & 2\,3\,4 & \ldots\ldots & 2n-1 & P\;Q\;R \\ n-2 & 1\,1\,1 & \ldots\ldots & 1 & n\;n\;n \end{pmatrix}$$
and $v = n$.

Rearranging the terms and writing them in terms of v, we have
$$T_{4v}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \ldots\ldots & 2v+2 & 2v+3 & 2v+4 & 2v+5 \\ 2v & 2v & 2v & 2v-2 & 2 & \ldots\ldots & 2 & 1 & 1 & 1 \end{pmatrix},$$
and Γ is
$$C_{2v}\begin{pmatrix} 1\,2\,3 & 4 & 5\,6 & \ldots\ldots & 2v+2 \\ v\;v\;v & v-2 & 1\,1 & \ldots\ldots & 1 \end{pmatrix}.$$

The fundamental curves are
$$\begin{pmatrix} 1\,2\,3 & 4 & 5\,6 & \ldots\ldots & 2v+2 & 2v+4 & 2v+5 \\ v\;v\;v & v-1 & 1\,1 & \ldots\ldots & 1 & 1 & 1 \end{pmatrix}^{1}_{2v}$$
$$\begin{pmatrix} 1\,2\,3 & 4 & 5 & \ldots\ldots & 2v+3 & 2v+5 \\ v\;v\;v & v-1 & 1 & \ldots\ldots & 1 & 1 \end{pmatrix}^{2}_{2v}$$
$$\begin{pmatrix} 1\,2\,3 & 4 & 5 & \ldots\ldots & 2v+4 \\ v\;v\;v & v-1 & 1 & \ldots\ldots & 1 \end{pmatrix}^{3}_{2v} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \ldots\ldots & 2v+2 \\ v-1 & v-1 & v-1 & v-2 & 1 & \ldots\ldots & 1 \end{pmatrix}^{4}_{2v+2}$$
$$(1\,2\,3\,4\,5)^{5}_{2} \ldots\ldots (1\,2\,3\,4\;2v+2)^{2v+2}_{2}\;(1\;2v+3)^{2v+3}_{1}\;(1\;2v+4)^{2v+4}_{1}$$
$$(1\;2v+5)^{2v+5}_{1}.$$

This transformation is equivalent to having in I three sets of three points on a line.

VII. Transform by $I_{\bar{P}QR}$, where $\bar{P}$ is on Γ and Q and R are not on Γ. Then

$$C_1\,I_{\bar{P}QR} = C_2\,(\bar{P}\;Q\;R).$$
$$C_2\,T_n = C_{2n}\begin{pmatrix} 1 & 2\,3\,4 & \ldots\ldots & 2n-1 & \bar{P}\;Q'\;R' \\ 2n-2 & 2\,2\,2 & \ldots\ldots & 2 & 1\;1\;1 \end{pmatrix},$$
$$C_{2n}\,I_{\bar{P}QR} = C_{4n-1}\begin{pmatrix} 1 & 2\,3 & \ldots\ldots & 2n-1 & \bar{P} & Q'' & R'' & Q & R \\ 2n-2 & 2\,2 & \ldots\ldots & 2 & 2n & 1 & 1 & 2n-1 & 2n-1 \end{pmatrix};$$
and Γ is
$$C_{2n-1}\begin{pmatrix} 1 & 2\,3\,4\,5 & \ldots\ldots & 2n-1 & \bar{P} & Q & R \\ n-2 & 1\,1\,1\,1 & \ldots\ldots & 1 & n & n-1 & n-1 \end{pmatrix}.$$
The class $v = n$.

The transformation in terms of v is
$$T_{4v-1}\begin{pmatrix} 1 & 2 & 3 & 4 & 5\,6 & \ldots\ldots & 2v+2 & 2v+3 & 2v+4 \\ 2v & 2v-1 & 2v-1 & 2v & 2\,2 & \ldots\ldots & 2 & 1 & 1 \end{pmatrix},$$
and Γ is
$$C_{2v-1}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \ldots\ldots & 2v+2 \\ v & v-1 & v-1 & v-2 & 1 & \ldots\ldots & 1 \end{pmatrix}.$$

The fundamental curves are

$$\begin{pmatrix}1 & 2 & 3 & 4 & 5 & \dots & 2v+4\\ v & v & v & v-1 & 1 & \dots & 1\end{pmatrix}^1_{2v} \begin{pmatrix}1 & 2 & 3 & 4 & 5 & \dots & 2v+3\\ v & v-1 & v-1 & v-1 & 1 & \dots & 1\end{pmatrix}^2_{2v-1}$$

$$\begin{pmatrix}1 & 2 & 3 & 4 & 5 & \dots & 2v+2 & 2v+4\\ v & v-1 & v-1 & v-1 & 1 & \dots & 1 & 1\end{pmatrix}^3_{2v-1}$$

$$\begin{pmatrix}1 & 2 & 3 & 4 & 5 & \dots & 2v+2\\ v-1 & v-1 & v-1 & v-2 & 1 & \dots & 1\end{pmatrix}^4_{2v-2}$$

$$(1\,2\,3\,4\,5)^5_2 \dots (1\,2\,3\,4\;2v+2)^{2v+2}_2\;(1\,2)^{2v+2}_1\;(1\,3)^{2v+3}_1.$$

It will be noticed that the point of intersection of the lines (14) and (23) is invariant; therefore the transformation is

$$T_{4v-1}\begin{pmatrix}1 & 2 & 3 & 4 & 5 & \dots & 2v+2 & 2v+3 & 2v+4 & 2v+5\\ 2v & 2v-1 & 2v-1 & 2v-2 & 2 & \dots & 2 & 1 & 1 & 0\end{pmatrix}.$$

This is equivalent to having four linear factors taken out of I.

VIII. Transform through $I_{2\bar{Q}\bar{P}}$, where $\bar{Q}$ and $\bar{P}$ are points on $\mathbf{\Gamma}$. Then

$$C_1\,I_{2\bar{Q}\bar{P}} = C_2\,(2\;\bar{Q}\;\bar{P}),$$

$$C_2\,T_n = C_{2n-1}\begin{pmatrix}1 & 2\,3\,4\,5 & \dots & 2n-1 & \bar{Q} & \bar{P}\\ 2n-3 & 1\,2\,2\,2 & \dots & 2 & 1 & 1\end{pmatrix},$$

$$C_{2n-1}\,I_{2\bar{Q}\bar{P}} = C_{4n-5}\begin{pmatrix}1 & 2 & 3 & \dots & 2n-1 & \bar{Q} & \bar{P}\\ 2n-3 & 2n-3 & 2 & \dots & 2 & 2n-3 & 2n-3\end{pmatrix}.$$

$\mathbf{\Gamma}$ is

$$C_{2n-3}\begin{pmatrix}1 & 2 & 3\,4 & \dots & 2n-1 & \bar{Q} & \bar{P}\\ n-2 & n-2 & 1\,1 & \dots & 1 & n-2 & n-2\end{pmatrix}.$$

Therefore $v = n-1$.

In terms of v, we have for the transformation

$$T_{4v-1}\begin{pmatrix}1 & 2 & 3 & 4 & 5 & \dots & 2v+3\\ 2v-1 & 2v-1 & 2v-1 & 2v-1 & 2 & \dots & 2\end{pmatrix},$$

while $\mathbf{\Gamma}$ is

$$C_{2v-1} = \begin{pmatrix}1 & 2 & 3 & 4\,5 & \dots & 2v+3\\ v-1 & v-1 & v-1 & 4\,1 & \dots & 1\end{pmatrix}.$$

The fundamental curves are

$$\begin{pmatrix}1 & 2 & 3 & 4\,5 & \dots & 2v+3\\ v-1 & v-1 & v-1 & v\,1 & \dots & 1\end{pmatrix}^1_{2v-1}$$

$$\begin{pmatrix}1 & 2 & 3 & 4 & 5 & \dots & 2v+3\\ v-1 & v-1 & v & v-1 & 1 & \dots & 1\end{pmatrix}^2_{2v-1}$$

$$\begin{pmatrix}1 & 2 & 3 & 4 & 5 & \dots & 2v+3\\ v-1 & v & v-1 & v-1 & 1 & \dots & 1\end{pmatrix}^3_{2v-1}$$

$$\begin{pmatrix}1 & 2 & 3 & 4 & 5 & \dots & 2v+3\\ v & v-1 & v-1 & v-1 & 1 & \dots & 1\end{pmatrix}^4_{2v-1}$$

$$(1\,2\,3\,4\,5)^5_2 \dots (1\,2\,3\,4\;2v+3)^{2v+3}_2.$$

It will be noticed that (1 2), (3 4), (1 3), (2 4) are invariant lines. This determines two invariant points. The transformation therefore is

$$T_{4v-1}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \; \ldots\ldots & 2v+3 & 2v+4 & 2v+5 \\ 2v-1 & 2v-1 & 2v-1 & 2v-1 & 2 \; \ldots\ldots & 2 & 0 & 0 \end{pmatrix},$$

which is equivalent to having three factors divided out of I.

IX. Transform by $I_{\bar{P}\bar{Q}R}$, where $\bar{P}$ and $\bar{Q}$ are on Γ and R is not. Then

$$C_1 I_{\bar{P}\bar{Q}R} = C_2 (\bar{P}\,\bar{Q}\,R),$$

$$C_2 T_n = C_{2n}\begin{pmatrix} 1 & 2\;3 \;\ldots\ldots & 2n-1 & \bar{P} & \bar{Q} & R' \\ 2n-2 & 2\;2 \;\ldots\ldots & 2 & 1 & 1 & 1 \end{pmatrix},$$

$$C_{2n} I_{\bar{P}\bar{Q}R} = C_{4n-2}\begin{pmatrix} 1 & 2\;3 \;\ldots\ldots & 2n-1 & \bar{P} & \bar{Q} & R'' & R \\ 2n-2 & 2\;2 \;\ldots\ldots & 2 & 2n-1 & 2n-1 & 1 & 2n-2 \end{pmatrix},$$

and Γ is

$$C_{2n-2}\begin{pmatrix} 1 & 2 \;\ldots\ldots & 2n-1 & \bar{P} & \bar{Q} & R \\ n-2 & 1 \;\ldots\ldots & 1 & n-1 & n-1 & n-1 \end{pmatrix}.$$

Therefore $v = n$ and the transformation in terms of v is

$$T_{4v-2}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \;\ldots\ldots & 2v+2 & 2v+3 \\ 2v-1 & 2v-1 & 2v-2 & 2v-2 & 2 \;\ldots\ldots & 2 & 1 \end{pmatrix}.$$

The fundamental curves are

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \;\ldots\ldots & 2v+3 \\ v-1 & v & v-1 & v-1 & 1 \;\ldots\ldots & 1 \end{pmatrix}^{1}_{2v-1}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \;\ldots\ldots & 2v+3 \\ v & v-1 & v-1 & v-1 & 1 \;\ldots\ldots & 1 \end{pmatrix}^{2}_{2v-1}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \;\ldots\ldots & 2v+2 \\ v-1 & v-1 & v-2 & v-1 & 1 \;\ldots\ldots & 1 \end{pmatrix}^{3}_{2v-2}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \;\ldots\ldots & 2v+2 \\ v-1 & v-1 & v-1 & v-2 & 1 \;\ldots\ldots & 1 \end{pmatrix}^{4}_{2v-2}$$

$$(1\,2\,3\,4\,5)^{5}_{2} \;\ldots\ldots\; (1\,2\,3\,4\;2v+2)^{2v+2}_{2}\;(1\,2)^{2v+3}_{1}.$$

The lines (1 3) (1 4) (2 4) (2 4) are invariant. Therefore there are two invariant points, and the transformation is

$$T_{4v-2}\begin{pmatrix} 1 & 2 & 3 & 4 & 5\;6 \;\ldots\ldots & 2v+2 & 2v+3 & 2v+4 & 2v+5 \\ 2v-1 & 2v-1 & 2v-2 & 2v-2 & 2\;2 \;\ldots\ldots & 2 & 1 & 0 & 0 \end{pmatrix}.$$

This is equivalent to dividing out five linear factors out of I.

X. Transform by $I_{\bar{P}\bar{Q}\bar{R}}$, where $\bar{P}\ \bar{Q}\ \bar{R}$ are points on Γ. Then

$$C_1 I_{\bar{P}\bar{Q}\bar{R}} = C_2\,(\bar{P}\ \bar{Q}\ \bar{R}),$$

$$C_2 T_n = C_{2n}\begin{pmatrix} 1 & 2\ 3 & \ldots . & 2n-1 & \bar{P}\ \bar{Q}\ \bar{R} \\ 2n-2 & 2\ 2 & \ldots . & 2 & 1\ 1\ 1 \end{pmatrix},$$

$$C_{2n} T_{\bar{P}\bar{Q}\bar{R}} = C_{4n-3}\begin{pmatrix} 1 & 2\ 3 & \ldots . & 2n-1 & \bar{P} & \bar{Q} & \bar{R} \\ 2n-2 & 2\ 2 & \ldots . & 2 & 2n-2 & 2n-2 & 2n-2 \end{pmatrix}.$$

Γ becomes

$$C_{2n-3}\begin{pmatrix} 1 & 2\ 3 & \ldots . & 2n-1 & \bar{P}\ Q\ \bar{R} \\ n-2 & 1\ 1 & \ldots . & 1 & n\ n\ n \end{pmatrix}.$$

Therefore $n = v$.

In terms of v, we have

$$T_{4v-3}\begin{pmatrix} 1 & 2 & 3 & 4 & 5\ 6 & \ldots . & 2v+2 \\ 2v-2 & 2v-2 & 2v-2 & 2v-2 & 2\ 2 & \ldots . & 2 \end{pmatrix}.$$

Γ becomes

$$C_{2v-3}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \ldots . & 2v+2 \\ v-2 & v-2 & v-2 & v-2 & 1 & \ldots . & 1 \end{pmatrix}.$$

The fundamental curves are

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \ldots . & 2v+2 \\ v-2 & v-1 & v-1 & v-1 & 1 & \ldots . & 1 \end{pmatrix}^{1}_{2v-2}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \ldots . & 2v+2 \\ v-1 & v-1 & v-1 & v-1 & 1 & \ldots . & 1 \end{pmatrix}^{2}_{2v-2}$$

$$\ldots .\ (1\ 2\ 3\ 4\ 5)^{5}_{2}\ \ldots .\ (1\ 2\ 3\ 4\ 2v+2)^{2v+2}_{2}.$$

But the lines (1 2) (1 3) (1 4) (2 3) (2 4) are invariant. Therefore there are three invariant points. The transformation therefore is

$$T_{4v-3}\begin{pmatrix} 1 & 2 & 3 & 4 & 5\ 6 & \ldots . & 2v+2 & 2v+3 & 2v+4 & 2v+5 \\ 2v-2 & 2v-2 & 2v-2 & 2v-2 & 2\ 2 & \ldots . & 2 & 0 & 0 & 0 \end{pmatrix}.$$

This is equivalent to dividing out six linear factors out of I.

By applying this process to any transformation which is involutorial, we can build involutorial transformations of all the different classes.

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